Recursive equations and basic properties of timed Petri nets

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Abstract

Timed Petri Nets provide a general formalism for describing the dynamics of Discrete Event Systems. The aim of this paper is to provide the basic equations that govern their evolution, when structural consumption conflicts are resolved by a predefined 'switching' mechanism. These equations can be seen as a non linear extension of the recursive equations for conflict free timed Petri nets, which are known to be linear in the $(\max, +)$ semi-field. These equations are shown to be "constructive" whenever the Petri net is live, and a computational scheme is given that allows one to determine the firing times of the transitions recursively. In the case of stochastic timed Petri nets, various structural properties are derived from these equations, including new stochastic monotony properties for certain queueing networks.

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Equations récursives et propriétés des réseaux de Petri temporisés

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Résumé

Les réseaux de Petri temporisés fournissent un formalisme général qui permet de décrire la dynamique des systèmes à événements discrets. L'objet de cet article est d'établir les équations de base qui régissent leur évolution, quand les conflicts entre transitions sont résolus par un mécanisme de routage prédéfini. Ces équations peuvent être vues comme une extension non linéaire des équations récursives pour des réseaux sans conflits, qui sont connues pour être linéaires dans le dioïde (max,+). On montre que les équations générales sont "constructives" quand le réseau est vivant et on donne un calcul algorithmique pour déterminer les instants de tir des transitions de façon itérative. Dans le cas de réseaux de Petri stochastiques, diverses propriétés structurelles se déduisent de ces équations, avec entre autres, de nouvelles propriétés de monotonie stochastique pour certains réseaux de file d'attente.

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1 Introduction

Timed Petri Networks can be viewed as a general formalism for describing the dynamics of Discrete Event Systems [10], [4]. In particular, similarly to other general formalisms like Matthes Schemas (see [7] and [2]), or Generalized Semi Markov Processes ([12], [13] and [8]), this formalism is powerful enough for allowing one to describe virtually all existing models in Queueing Theory.

The aim of this paper is to provide the basic equations that govern the evolution of such a network, when taking for state variables the epochs at which events occur. The first attempt to derive such equations for general timed Petri nets can be traced back to [5]. However, the approach proposed in that paper did not lead to explicit equations, due to the problem of consumption conflicts. To the best of the authors knowledge, this type of equations are only available for specific classes of Petri Nets, like Event Graphs (see [6] for the case with constant timing, and [1] for the non-constant case). The approach that is proposed in the present paper can be seen as a continuation of these last two papers. It is based on a resolution of the consumption conflicts which is based on a pre-defined routing mechanism attached to tokens. It essentially differs from the classical literature on timed and stochastic Petri nets (see [11], [9]) in that the state variables that are considered are *not* the (integer valued) marking.

The definition of the class of Petri nets of interest is given in section 3 and section 4. It is shown in section 5 that this predefined switching mechanism allows one to derive an evolution equation for any timed FIFO Petri net with constant or variable timing sequences. This equation can be seen as a non linear extension of the $(\max, +)$-linear recursive equation that is known to hold for timed Event Graphs. The "integral solution" of the recursive equation is shown to be compatible with the class of $(\min, \max)$ solutions introduced independently by Glasserman and Yao for Matthes Schemas in [14] (see section 6). In section 7, we prove the 'constructiveness' of the recursive equation in the case where the Petri net under consideration is live. Finally, section 8 focuses on various monotony properties of the state variables that follow from these equations. These monotony properties are shown to have interesting implications in queueing theory.

2 Petri Nets

2.1 Definition

A Petri Net is characterized by a triple (P, T, Γ), where P is the set of places, T the set of transitions and $\Gamma = (V, E)$ a directed graph with vertices $V = P \bigcup T$ and edges E. The edges of Γ are either of the form (p, t) or of the form (t, p) with $p \in P$ and $t \in T$, that is Γ is a bipartite graph. We shall denote

- $\gamma^-(p)$ the set of transitions that precede place p in Γ: $\gamma^-(p) = \{t \in T \mid (t, p) \in E\}$;
- $\gamma^+(p)$ the set of transitions that follow place p in Γ: $\gamma^+(p) = \{t \in T \mid (p, t) \in E\}$;
- $\gamma^-(t)$ the set of places that precede transition t in Γ: $\gamma^-(t) = \{p \in P \mid (p, t) \in E\}$;
- $\gamma^+(t)$ the set of places that follow transition t in Γ: $\gamma^+(t) = \{p \in P \mid (t, p) \in E\}$.

Tokens circulate in the Petri Net. This circulation takes place when transitions are fired. Two functions r and $s : P \times T \to I\!N$ are used to describe this mechanism:

- $r_{p,t}$, the so-called backward-incidence function, gives the number of tokens that are consumed by the firing of transition t in place $p \in \gamma^-(t)$;
- $s_{p,t}$, the so-called forward-incidence function, gives the number of tokens that are produced by the firing of transition t in place $p \in \gamma^+(t)$.

Roughly speaking, the evolution of a Petri Net (P, T, Γ), of incidence functions r and s, is specified by the following rules: an initial condition is first given, the so-called initial marking, which specifies the number of tokens M_p initially present in place p. A transition t is said to be enabled if for each place p of $\gamma^-(t)$ there are at least $r_{p,t}$ tokens present in the place. Once enabled, a transition can fire. The firing of transition t consumes exactly $r_{p,t}$ tokens in place p for all $p \in \gamma^-(t)$ and produces $s_{p,t}$ tokens in place p for all $p \in \gamma^+(t)$.

2.2 FIFO Timed Petri Nets (TPN)

We shall adopt the following definition concerning the numbering of tokens traversing a place and the numbering of firings of a transition:

- the initial tokens of place p are numbered $1, \ldots, M_p$, while the n-th token, $n > M_p$, of place p is the $(n - M_p)$-th to *enter* p after the beginning of the network evolution. Tokens entering p at the same time are numbered arbitrarily.
- The n-th firing, $n \geq 1$, of transition t is the n-th firing of t to be enabled from the beginning of the network evolution. Firings of t enabled at the same time are numbered arbitrarily (within our formalism, nothing prevents the same transition to be enabled twice at the same epoch).

Timing is involved in the evolution of the system through the following two rules:

- The n-th initial token of place p, $n \leq M_p$, is not considered immediately available for downstream transitions. It is put in place p at time $v_p(n)$ (where the function v_p is given), and it has then to stay in p for a minimal *holding time* $\sigma_p(n)$ before being considered available to enable the transitions that follows p. Similarly, the n-th token of place p, $n > M_p$ (or equivalently the $(n - M_p)$-th to enter p) can only be taken into account by the transitions that follows p, $\sigma_p(n)$ units of time after its arrival.
- Each transition starts firing as soon as it is enabled (we shall discuss the problem that arises with conflicts later on). Once transition t is enabled for the n-th time, the tokens that it intends to consume become reserved tokens (they cannot contribute to enabling another transition before being consumed by the firing of transition t). Once it is enabled, the time for transition t to complete its n-th firing takes the *firing time* $\sigma_t(n)$. Once the firing time is completed, the transition completes its firing. This firing completion consists in withdrawing $r_{p,t}$ tokens from each of the

places that precede t (the reserved tokens), and producing $s_{p,t}$ new tokens into the places that follow t. These two actions are supposed to be simultaneous.

We now define what FIFO places and transitions are. Both places and transitions are seen as input/output systems. For instance, a transition has a sequence of enabling times (when it starts firing) as input and a sequence of completion times (when it completes firing and sends a bulk of tokens in each of the places downstream).

- A place is FIFO if tokens become available in this place in the same order as they reached it.
- A transition t is said to be FIFO if the n-th firing completion of that transition corresponds to the n-th firing that t has started.

If the rule is that each transition consumes tokens as they become available, a place is FIFO if and only if tokens cannot overtake one another in this place due to holding times (which is ensured for instance when these holding times are constant). A transition is FIFO in the particular case where the topology of the Petri Net is such that the $(n+1)$-st firing of this transition can only start after the completion of the n-th firing (which happens when this transition t is recycled, *i.e.* we add a place p with one initial token and the edges (p,t) and (t,p)).

If the holding and firing times, and the initial delays are random variables defined on some probability space, then the timed Petri Net under consideration is said to be a Stochastic Petri Net.

3 Recursive Equations

Let $I\!R$ denote the set of real numbers and $I\!N$ denote the set of nonnegative integers. Let $Z = (Z_1, \cdots, Z_n) = (Z_m)_{m=1,n}$ a vector of $(I\!R)^n$. We denote by $\mathcal{R}(Z_m)_{m=1,n}$ the vector

$$\mathcal{R}(Z_m)_{m=1,n} = (Z_{i(1)}, \cdots, Z_{i(n)}) \in I\!R^n,$$

where $i : \{1, \cdots, n\} \to \{1, \cdots, n\}$ is a bijection such that

$$Z_{i(1)} \leq Z_{i(2)} \cdots \leq Z_{i(n)}.$$

This notation is extended to vectors of $(I\!R)^{I\!N}$, whenever meaningful.

3.1 State Variables

Let $X_t(n),\ t \in T,\ n \geq 1$ denote the time when transition t starts firing for the n-th time, with the convention that for all $t \in T$, $X_t(n) = \infty$ if transition t never fires for the n-th time.

Let $Y_t(n),\ t \in T,\ n \geq 1$ denote the time when transition t completes its n-th firing, with the convention that for all $t \in T$, $Y_t(n) = \infty$ if transition t never fires for the n-th time.

Let $V_p(n),\ p \in P,\ n \geq 1$ denote the time when place p receives its n-th token, with the convention that for all $p \in P$, $V_p(n) = \infty$ if the place never receives its n-th token.

Let $W_p(n),\ p \in P,\ n \geq 1$ denote the time when place p releases its n-th token, with the convention that for all $p \in P$, $W_p(n) = \infty$ if the n-th token is never released.

As to holding and firing times, observe that due to our conventions, $\sigma_t(n)$ denotes the firing time of t that starts at $X_t(n)$, $n \geq 1$, while $\sigma_p(n)$ denotes the holding time of the token that enters p at $V_p(n)$, $n \geq 1$.

If transition t is FIFO, we have the obvious relation

$$Y_t(n) = X_t(n) + \sigma_t(n). \tag{3.1}$$

More generally,

$$(Y_t(n))_{n\geq 1} = \mathcal{R}((X_t(n) + \sigma_t(n))_{n\geq 1}). \tag{3.2}$$

If place p is FIFO, we can write

$$W_p(n) \geq V_p(n) + \sigma_p(n), \tag{3.3}$$

since the token that enters p at time $V_p(n)$ stays there for at least $\sigma_p(n)$. More generally,

$$(W_p(n))_{n\geq 1} \geq \mathcal{R}((V_p(n) + \sigma_p(n))_{n\geq 1}). \tag{3.4}$$

3.2 Initial Conditions

It is assumed that an origin of time and the initial marking have been fixed in such a way that the variables $V_p(n)$ and $W_p(n)$ satisfy the bounds

$$V_p(n) \begin{cases} = v_p(n) < 0, & \text{for } n = 1, \cdots, M_p, \text{ if } M_p \geq 1; \\ \geq 0, & \text{for } n > M_p, \end{cases} \tag{3.5}$$

and

$$W_p(n) \geq 0, \quad \text{for } n \geq 1. \tag{3.6}$$

These conventions are natural: they mean that tokens arrived in place p prior to the initial time and which left p before that initial time are not considered to belong to the initial marking. Similarly, tokens arrived in p "at or after" the initial time do not belong to the initial marking.

3.3 Upstream Equations Associated with Transitions

We first look at the relationships induced by a transition t due to the places preceding t. We first consider the case without *structural consumption conflicts*, namely, for every place p preceding t, the set of transitions that follow p is reduced to t (this restriction corresponds to the class of decision free Petri nets also known as Event Graphs in the literature).

3.3.1 No Structural Consumption Conflicts

For all $p \in \gamma^-(t)$, $r_{p,t}$ tokens leave place p at $W_p(n)$. Since t is the only transition that can consume the tokens of p, this corresponds to the starting of the n-th firing of t. Hence

$$X_t(n) = W_p((n-1)r_{p,t}+1) = \cdots = W_p(nr_{p,t}), \quad \forall p \in \gamma^-(t). \tag{3.7}$$

In the FIFO case, the last token of place p to become available for enabling t for the n-th time must be the $nr_{p,t}$-th to enter p, so that

$$X_t(n) = \max_{p \in \gamma^-(t)} \{V_p(nr_{p,t}) + \sigma_p(nr_{p,t})\}. \tag{3.8}$$

More generally,

$$X_t(n) = \max_{p \in \gamma^-(t)} Z_p(nr_{p,t}), \tag{3.9}$$

where

$$(Z_p(n))_{n\geq 1} = \mathcal{R}((V_p(n) + \sigma_p(n))_{n\geq 1}). \tag{3.10}$$

3.3.2 General Case

Without further specifications on how the conflict is resolved, we can only state the following inequalities: in the FIFO case

$$X_t(n) \geq \max_{p \in \gamma^-(t)} \{V_p(nr_{p,t}) + \sigma_p(nr_{p,t})\} \tag{3.11}$$

and more generally

$$X_t(n) \geq \max_{p \in \gamma^-(t)} Z_p(nr_{p,t}). \tag{3.12}$$

These inequalities are not very satisfactory, and we shall come back to this point later on.

3.4 Downstream Equations Associated with Transitions

We now look at the relationships induced by a transition t due to the places following t. We first consider the case without *structural supply conflicts*, namely, for every place p following t, the set of transitions that precede p is reduced to t (that is, p is only fed by this t).

3.4.1 No Structural Supply Conflicts

If no other transition than t can feed the places following t, the tokens arriving at place $p \in \gamma^+(t)$ in rank $M_p + (n-1).s_{p,t} + 1$ to $M_p + n.s_{p,t}$ have been produced bh the n-th firing of transition t, therefore

$$Y_t(n) = V_p(M_p + (n-1)s_{p,t} + 1) = \cdots = V_p(M_p + ns_{p,t}), \quad \forall p \in \gamma^+(t), \tag{3.13}$$

In the FIFO case, this leads to the relation

$$X_t(n) + \sigma_t(n) = V_p(M_p + (n-1)s_{p,t} + 1) = \cdots = V_p(M_p + ns_{p,t}), \quad \forall p \in \gamma^+(t), \tag{3.14}$$

while in the general case

$$\mathcal{R}((X_t(k) + \sigma_t(k))_{k\geq 1})_n = V_p(M_p + (n-1)s_{p,t} + 1) = \cdots = V_p(M_p + ns_{p,t}), \quad \forall p \in \gamma^+(t). \tag{3.15}$$

3.4.2 General Case

Without further specifications, we can only state the following inequalities: in the FIFO case

$$X_t(n) + \sigma_t(n) \geq V_p(M_p + ns_{p,t}) \geq \cdots \geq V_p(M_p + (n-1)s_{p,t} + 1) \quad \forall p \in \gamma^+(t), \tag{3.16}$$

while in the general case

$$\mathcal{R}((X_t(k) + \sigma_t(k))_{k\geq 1})_n \geq V_p(M_p + ns_{p,t}) \geq \cdots \geq V_p(M_p + (n-1)s_{p,t} + 1) \quad \forall p \in \gamma^+(t). \tag{3.17}$$

3.5 Upstream Equations Associated with Places

We now focus our attention on the upstream relationships induced by a place p. Consider the sequences $(Y_t(n))_{n\geq 1}$, for all $t \in \gamma^-(p)$. With each of them, associate a point process on the real line, where the points are located at $Y_t(n)$, each with multiplicity $s_{p,t}$. We can look at the arrival process into p as the superposition of these $|\gamma^-(p)|$ point processes.

With all $t \in \gamma^-(p)$, we associate an integer $i_t \in I\!N$ representing a number of complete firings of t. If transition t has completed exactly i_t firings for all $t \in \gamma^-(p)$, then place p has received exactly

$$\sum_{t\in\gamma^-(p)} i_t s_{p,t}$$

tokens. The set of vectors $(i) = (i_t,\ t \in \gamma^-(p))$ such that the n-th token has entered place p is hence

$$\mathcal{A}_n^p = \left\{ i \in I\!N^{|\gamma^-(p)|} \mid \sum_{t\in\gamma^-(p)} i_t s_{p,t} \geq n \right\}.$$

Observe that we cannot in general replace $\geq n$ by $= n$ in the last definition because of the bulk arrivals. The last token produced by the transition firings specified by some $i \in \mathcal{A}_n^p$ arrives in p at time

$$\max_{t\in\gamma^-(p)} Y_t(i_t),$$

where $Y_t(0) = 0$ by convention. Since at least n tokens have reached p once all the firings specified by i have been completed, one gets

$$V_p(n + M_p) \leq \inf_{i\in\mathcal{A}_n^p} \{ \max_{t\in\gamma^-(p)} Y_t(i_t)\}.$$

On the other hand,

$$V_p(n + M(p)) > \sup_{i\notin\mathcal{A}_n^p} \{ \max_{t\in\gamma^-(p)} Y_t(i_t)\},$$

since from the very definition of $\mathcal{A}_n^p$, the collection of events represented by $i \notin \mathcal{A}_n^p$ brings strictly less that n tokens in p. But $V_p(n + M_p)$ should be equal to some $Y_{t_0}(n_0)$

since at least one collection of events puts n tokens in place p (unless $\mathcal{A}_n^p$ is empty and $V_p(n+M_p)=\infty$). Hence equality must hold true in $V_p(n+M_p)=\inf\ldots$. We get then the following final relation

$$V_p(n+M_p)=\inf_{\left\{i\in I\!N^{|\gamma^-(p)|}\ \mid\ \sum_{t\in\gamma^-(p)} i_t s_{p,t}\geq n\right\}}\left\{\max_{t\in\gamma^-(p)} Y_t(i_t)\right\}, \tag{3.18}$$

where $Y_t(0)=0$ by convention.

3.6 Downstream Equations Associated with Places

We concentrate now on the downstream relationships induced by a place p. It is in this type of equations that the *structural consumption conflicts* associated with general Petri Nets become apparent.

Consider the sequences $(X_t(n))_{n\geq 1}$, for all $t\in\gamma^+(p)$. With all $t\in\gamma^+(p)$, we associate an integer $i_t\in I\!N$ representing some number of firing initiations of t. If t has started exactly i_t firings for all $t\in\gamma^+(p)$, then exactly

$$\sum_{t\in\gamma^+(p)} i_t r_{p,t}$$

tokens have been withdrawn from p. The set of vectors $i=(i_t,\ t\in\gamma^+(p))$ such that the n-th token has left place p is hence

$$\mathcal{B}_n^p=\left\{i\in I\!N^{|\gamma^+(p)|}\ \Big|\ \sum_{t\in\gamma^+(p)} i_t r_{p,t}\geq n\right\}.$$

Again, we cannot in general replace $\geq n$ by $=n$. For any i in this set, the last token to leave p leaves at time

$$\max_{t\in\gamma^+(p)} X_t(i_t).$$

Hence

$$W_p(n) \leq \inf_{i \in \mathcal{B}_n^p} \{ \max_{t \in \gamma^+(p)} X_t(i_t) \}.$$

Using a similar reasoning as previously, we get the final relation

$$W_p(n) = \inf_{\left\{ i \in \mathbb{N}^{|\gamma^+(p)|} \;|\; \sum_{t \in \gamma^+(p)} i_t r_{p,t} \geq n \right\}} \left\{ \max_{t \in \gamma^+(p)} X_t(i_t) \right\}. \tag{3.19}$$

Relations (3.18) and (3.19) exhibit nothing more than a superficial symmetry. Indeed, while (3.18) allows one to construct the sequence $V_p(n)$ from the knowledge of what happens upstream p and earlier, this is not true at all for (3.19) that only provides some sort of *backward* property stating that the knowledge of what will happen following p in the future allows one to reconstruct what happens in p now.

The reason for this is that the way the conflict is solved is not yet sufficiently precise. We show now one natural way of solving conflicts, that we will call *switching*. Several other ways are conceivable like *competition* that we will also outline.

3.6.1 Switching

Within this setting, each place that has several transitions downstream receives a switching sequence $\rho_p(n)$ with values in $\gamma^+(p)^{\mathbb{N}}$. In the same way as the n-th token to enter place p receives a holding time $\sigma_p(n)$, it also receives a route to which to be switched. This information is given by $\rho_p(n)$, which specifies which transition it should be routed to. In other words, only those tokens such that $\rho_p(n) = t$ should be taken into account by $t \in \gamma^+(p)$. The function ρ_p is said to be compatible with all the backward incidence functions $r_{p,t}$ for $t \in \gamma^+(p)$ in the sense that if $\rho_p(n) \neq t$ and $\rho_p(n+1) = t$ then $\rho_p(n+k) = t$ for all $1 \leq k \leq r_{p,t}$. In case the place p is FIFO, the non-compatibility of ρ_p with the backward incidence functions leads to a deadlock of p. In the following, the switching will be assumed to be compatible. In the case system under consideration is a stochastic Petri net, these switching variables are also assumed to be random variables defined on the same probability space as the holding and firing times. By doing so, one completely specifies the behavior of the system. For instance, in the FIFO case, one gets the inequality

$$X_t(n) \geq W_p(\eta_{p,t}(nr_{p,t})), \quad \forall p \in \gamma^-(t),$$

where the *switching function* $\eta_{p,t}$ is defined by

$$\eta_{p,t}(n) = \inf \left\{ m \geq 1 \mid \sum_{k=1}^{m} 1\{\rho_p(k) = t\} \geq n \right\}. \tag{3.20}$$

Whenever the behaviors of the other places upstream t are specified, one can go further and get the desired *forward* equation, as we will see in the next Section.

3.6.2 Competition

The places following p compete for the tokens of p on a First Come First Serve (FCFS) basis: within this interpretation, the tokens that have been served in place p can be seen as building up some *queue of tokens*. Once a transition t following p is enabled except for the condition depending on p, it puts some request for $s_{p,t}$ tokens in some FCFS queue of requests. This request is served (and the corresponding transition enabled) as soon as it is at the head of the request line and there are $s_{p,t}$ tokens in the token queue.

4 Recursive Equations for Switching

In this section, it is assumed that all places receive some switching. For places with a single downstream transition, this sequence is trivial in the sense that it always routes tokens to this transition.

Theorem 1 *Under the foregoing assumptions, the state variables $V_p(n)$, $p \in P$, $n \geq 1$ of a FIFO TPN satisfy the recursive equations*

$$V_p(n + M_p) \quad = \quad \inf_{\left\{ i \in \mathbb{N}^{|\gamma^-(p)|} \mid \sum_{t \in \gamma^-(p)} i_t s_{p,t} \geq n \right\}} \quad \max_{\{t \in \gamma^-(p),\ q \in \gamma^-(t)\ \}}$$

$$\{V_q(\eta_{q,t}(i_t r_{q,t})) + \sigma_q(\eta_{q,t}(i_t r_{q,t})) + \sigma_t(i_t)\}, \quad n \geq 1, \tag{4.1}$$

with the initial condition $V_p(n) = v_p(n)$ *for* $1 \leq n \leq M_p$*, if* $M_p \geq 1$.

Proof Besides the variables $V_p(n)$, we will also make use of the auxiliary variables $X_t(n)$, $t \in T$, $n \geq 1$. Owing to the switching assumptions, inequality (3.11) can be replaced by the relation

$$X_t(n) = \max_{\{p \in \gamma^-(t)\}} \inf_{\{ j_p \geq 1 \mid \sum_{k=1}^{j_p} 1\{\rho_p(k)=t\} \geq n r_{p,t}\}} \{V_p(j_p) + \sigma_p(j_p)\}, \quad n \geq 1 \tag{4.2}$$

or equivalently

$$X_t(n) = \max_{p \in \gamma^-(t)} \{V_p(\eta_{p,t}(n r_{p,t})) + \sigma_p(\eta_{p,t}(n r_{p,t}))\}, \quad n \geq 1, \tag{4.3}$$

where we used the routing function $\eta_{p,t}$ defined in (3.20), and the FIFO assumption, which implies that the mapping $i \rightarrow V_p(i) + \sigma_p(i)$ is nondecreasing.

Similarly using (3.1) in (3.18) yields

$$V_p(n + M_p) = \inf_{\left\{i \in I\!N^{|\gamma^-(p)|} \mid \sum_{t \in \gamma^-(p)} i_t s_{p,t} \geq n \right\}} \max_{\{t \in \gamma^-(p)\}} \{X_t(i_t) + \sigma_t(i_t)\}, \quad n \geq 1. \tag{4.4}$$

Equation (4.1) follows immediately from (4.3) and (4.4).

Remark 1
In (4.1), the infinimum over the infinite set

$$\left\{ i \in I\!N^{|\gamma^-(p)|} \mid \sum_{t \in \gamma^-(p)} i_t s_{p,t} \geq n \right\} \tag{4.5}$$

can be replaced by a minimum over the finite set

$$\left\{ i \in I\!N^{|\gamma^-(p)|} \mid n \leq \left(\sum_{t \in \gamma^-(p)} i_t s_{p,t} \right) \leq n + \max_{t \in \gamma^-(p)} s_{p,t} \right\}. \tag{4.6}$$

Remark 2

In the case where the TPN is not FIFO, Equations (4.3) and (4.4) have to be replaced by

$$X_t(n) = \max_{p\in\gamma^-(t)} \left\{ (\mathcal{R}(V_p(m) + \sigma_p(m))_{m\geq 1})_{(\eta_{p,t}(nr_{p,t}))} \right\}, \quad n \geq 1 \tag{4.7}$$

and

$$V_p(n+M_p) = \inf_{\left\{ i\in\mathbb{N}^{|\gamma^-(p)|} \;\mid\; \sum_{t\in\gamma^-(p)} i_t s_{p,t} \geq n \right\}} \max_{\{t\in\gamma^-(p)\}} \left\{ (\mathcal{R}(X_t(m) + \sigma_t(m))_{m\geq 1})_{(i_t)} \right\}, \quad n \geq 1 \tag{4.8}$$

respectively.

Remark 3

In (4.1), we can get rid of the firing times $\sigma_t(n)$ by changing the holding times $\sigma_p(\eta_{p,t}(n))$, $\forall p \in \gamma^-(t)$ into $\sigma_p(\eta_{p,t}(n)) + \sigma_t(n)$. Thus we get an equivalent net with $\sigma_t(n) = 0$ and $\sigma_p(n) > 0,\ t \in T,\ p \in P,\ n \geq 1$ where the equivalence means that the entrance times are the same in both systems.

5 Integration of the Recursive Equations

In this section, we use a tree structure to get an "integration" of the recursive equations. Analog results have been obtained in [[14]] using language theory.

5.1 Evolution Trees

¿From now on, we introduce a new structure, the trees. A tree is a directed, connected, acyclic, valuated graph with only one source (*i.e.* with a unique vertex with no incident edge). A specific vocabulary is used for these graphs. The vertices are called the *nodes* of the tree, the source is called the *root* of the tree, the sinks (*i.e.* the vertices with no outgoing edges) are called the *leaves* of the tree and the valuations of the nodes are called the *weights* of the nodes. We call the *weight* of a directed path the sum of the weights of all its nodes but its source. A node N_1 is said to be *deeper* than a node N_2 if we can find a directed path from N_2 to N_1. Finally, the *depth* of a tree is the length of its longest directed path.

Definition 1 *Let $(p,n) \in P \times IN$. We define an Evolution Tree A of (p,n) . If $n \leq M_p$ then A is reduced to a single node (p,n) with weight $\sigma_p(n) + v_p(n)$. If $n > M_p$, let $i \in IN^{|\gamma^-(p)|}$ satisfy:*

$$\sum_{t \in \gamma^-(p)} i_t = n - M_p \ .$$

An Evolution Tree A associated with (p,n) is obtained by hanging to the node (p,n) one (among the several possible) Evolution Tree associated with each node $(q, \eta_{q,t}(i_t))$, $t \in \gamma^-(p)$, $q \in \gamma^-(t)$. The nodes corresponding to the roots of the latter trees are respectively given a weight $\sigma_q(\eta_{q,t}(i_t))$. The root (p,n) has a weight $\sigma_p(n)$.

We will denote by $\mathcal{E}(p,n)$ the set of all the Evolution Trees of the couple (p,n).

5.2 Integration of the Recursive Equations

In the following, we assume that the valuations of the arcs of the net are all 1 (i.e. $\forall(t,p) \in \Gamma, \ \ s_{t,p} = 1$ and $\forall(p,t) \in \Gamma, \ \ r_{p,t} = 1$). We also assume that the TPN is FIFO. We use Remark 3 to get $\sigma_t(n) = 0, \ \ t \in T, \ n \geq 1$. Finally, we assume that the swichting is given as well as the holding times in the places and that on every cycle of the Petri net, there is a place p with $0 < \sigma_p(n) < \infty \ \ , \ \ n \geq 1$.

In equation (4.1) we can replace the variables $V_q(\eta_{q,t}(n))$ by using once more equation (4.1). We get:

$$V_p(n+M_p) = \min_{\left\{ i\in \mathbb{N}^{|\gamma^-(p)|} \mid \sum_{t\in\gamma^-(p)} i_t = n \right\}} \max_{\{t\in\gamma^-(p),\ q\in\gamma^-(t)\ \}}$$

$$\min_{\left\{ j\in \mathbb{N}^{|\gamma^-(q)|} \mid \sum_{s\in\gamma^-(q)} j_s = \eta_{q,t}(i_t)-M_q \right\}} \max_{\{s\in\gamma^-(q),\ r\in\gamma^-(s)\ \}}$$

$$\{V_r(\eta_{r,s}(j_s)) + \sigma_r(\eta_{r,s}(j_s)) + \sigma_q(\eta_{q,t}(i_t))\}\,.$$

Using the distributivity of max with respect to min, this equality becomes:

$$V_p(n+M_p) = \min_{\left\{ \begin{array}{c} i\in \mathbb{N}^{|\gamma^-(p)|} \mid \sum_{t\in\gamma^-(p)} i_t = n \\ j^{q,t}\in \mathbb{N}^{|\gamma^-(q)|}\ ,\ t\in\gamma^-(p),\ q\in\gamma^-(t) \mid \sum_{s\in\gamma^-(q)} j_s^{q,t} = \eta_{q,t}(i_t)-M_q \end{array} \right\}} \max_{\left\{ \begin{array}{c} t\in\gamma^-(p) \\ q\in\gamma^-(t) \\ s\in\gamma^-(q) \\ r\in\gamma^-(s) \end{array} \right\}}$$

$$\left\{V_r(\eta_{r,s}(j_s^{q,t})) + \sigma_r(\eta_{r,s}(j_s^{q,t})) + \sigma_q(\eta_{q,t}(i_t))\right\}. \tag{5.1}$$

Equation (5.1), represents the first step in the "integration" of the recursive equations. Indeed, we obtain a tree of depth 2 from the root $(p, n+M_p)$. If we continue to develop this equation, we get trees with increasing depths. We stop when each path ends with a leaf, namely, when it terminates with a node (q,m) with $m \leq M_q$. We eventually obtain the integration of the Equation (4.1).

$$V_p(n) = \inf_{A\in\mathcal{E}(p,n)} \mathcal{C}(A)\ ,\quad n \geq M_p, \tag{5.2}$$

where :

$$\mathcal{C}(A) = \sup_{T\in\mathcal{T}(A)} (w(T)), \tag{5.3}$$

is the cost of tree A,

$\mathcal{T}(A)$ is the set of all the directed paths from the root to any leaf of the tree A and $w(T)$ is the weight of the directed path T (*i.e.* the sum of the weights of all its nodes except its root).

The set $\mathcal{E}(p,n)$ might contain infinite trees, thus $\mathcal{E}(p,n)$ is not constructible and this transformation of the recursive equations does not obviously give the "constructiveness" caracter of these equations. However, it is useful for preliminary results.

5.3 Preliminary Properties.

In $\mathcal{E}(p,n)$, some trees are of great interest. These trees are called the Real Evolution Trees (RET) and are defined recursively as follows. If $n \leq M_p$, then the node (p,n) admits only one RET, R reduced to a single node (p,n) with weight $\sigma_p(n) + v_p(n)$. If $n > M_p$, choose $i \in I\!N^{|\gamma^-(p)|}$ such that:

$$\sum_{t \in \gamma^-(P)} i_t = n - M_p$$

and

$$V_p(n) = \max_{\{t \in \gamma^-(p), q \in \gamma^-(t)\}} \{V_q(\eta_{q,t}(i_t)) + \sigma_q(\eta_{q,t}(i_t))\}.$$

Then define a RET R of (p,n) by hanging underneath the node (p,n), one Real Evolution Tree of each couple $(q, \eta_{q,t}(i_t))$, $t \in \gamma^-(p)$, $q \in \gamma^-(t)$. The nodes corresponding to the roots of the latter trees are respectively given a weight $\sigma_q(\eta_{q,t}(i_t))$. The root (p,n) has a weight $\sigma_p(n)$.

This construction induces the following straightforward properties: $\mathcal{C}(R) = V_p(n)$ and if R' is any subtree of R, with a root (q,m) then, R' is a RET of the node (q,m) and $\mathcal{C}(R') = V_q(m)$.

Definition 2 *The switching of a TPN is said to be "fair" if*

$$\eta_{p,t}(n) < \infty, \quad \forall p \in P,\ t \in \gamma^+(p),\ n \geq 1.$$

We recall that a TPN is live if all its transitions will fire infinitely often. We see that a switched TPN cannot be live unless its switching is weakly fair.

Indeed, assume the switching is not fair. Then we can find $p \in P$, $t \in \gamma^+(p)$, $n \geq 1$, with $\eta_{p,t}(n) = \infty$. Now, transition t needs to wait for the $\eta_{p,t}(n)$-th token of place p to be able to fire for the n-th time. Therefore the TPN is not live.

In the following, the switching is always assumed to be fair.

Lemma 1 *A TPN is live if and only if* $V_p(n) < \infty, \quad \forall p \in P, \ n \geq 1$.

Proof First, assume that the TPN is live. Let $p \in P$, $t \in \gamma^-(p)$. Let $(n_i)_{i \in I\!N}$ be the indices of the tokens coming from t into p. Since the TPN is live, $(n_i)_{i \in I\!N}$ is an infinite increasing sequence. Let n be any integer greater than 1. We can choose i such that $n_i \geq n$. Now $V_p(n) \leq V_p(n_i)$, because the net is FIFO and $V_p(n_i) < \infty$, since the net is live.

Conversely, let $X_t(n)$ be the time when transition t fires for the n-th time. Obviously, the TPN is live if and only if $X_t(n) < \infty, \quad t \in T, \ n \geq 1$. Since the TPN is FIFO, we get for $t \in T$ and $n \in I\!N$:

$$X_t(n) = \max_{p \in \gamma^-(t)} \{V_p(\eta_{p,t}(n)) + \sigma_p(\eta_{p,t}(n))\}.$$

Since the net is fair, for all $p \in \gamma^-(t)$ we have $\eta_{p,t}(n) < \infty$ and if we assume that $V_p(m) < \infty$,$\forall m \in I\!N$, we also get $V_p(\eta_{p,t}(n)) < \infty$. Finally, we know by convention that $\sigma_p(\eta_{p,t}(n)) < \infty$ and we obtain $X_t(n) < \infty$.

Definition 3 *A tree A in $\mathcal{E}(p,n)$ is "anticipative" if there exist two nodes $(q,k),(q,m)$ in A, (q,m) being deeper than (q,k), with $k \leq m$.*

Thanks to this definition we can find a sufficient condition for the TPN to be live, based on the Evolution Trees.

Lemma 2 *If a TPN is live, then let $R \in \mathcal{E}(p,n)$ be a Real Evolution Tree. Then R is non-anticipative.*

Proof Suppose R is anticipative. We can find two nodes $(q,k),(q,m)$ in R, (q,m) being deeper than (q,k), with $k \leq m$. We now consider the subtrees R_1 and R_2 of R with respective roots (q,k) and (q,m). We know that $\mathcal{C}(R_1) = V_q(k)$ and $\mathcal{C}(R_2) = V_q(m)$. Since the TPN is FIFO, we have the inequality $V_q(m) \geq V_q(k)$. On the other hand, since (q,m) is deeper than (q,k), we have $\mathcal{C}(R_1) \geq \mathcal{C}(R_2)$, because the weight of every node in R is non-negative. Therefore, $V_q(m) = V_q(k)$. Furthermore, the path from the node (q,k) to the node (q,m) in the tree is denoted $(p_1,m_1),(p_2,m_2),\cdots,(p_j,m_j)$ and this path is induced by a cycle in the Petri network : $p_1,p_2,\cdots,p_j$. In this cycle we have $\sigma_{p_i}(m_i) = 0$ for $1 \leq i \leq j$. This contradicts the hypothesis on the holding times.

Lemma 3 *If there exists a tree A in $\mathcal{E}(p,n)$ for all $p \in P$, $n \geq 1$ which is non-anticipative, then the TPN is live.*

Proof If A is non-anticipative, then A is a finite tree. Therefore, $\mathcal{C}(A) < \infty$. Now, $V_p(n) \leq \mathcal{C}(A)$ allows to conclude the proof.

Theorem 2 *The TPN is live if and only if there exists a tree A in $\mathcal{E}(p,n)$ for all $p \in P$, $n \geq 1$ which is non-anticipative.*

Proof This result is a direct consequence of the two preceding lemmas.

Corollary 1 *The liveness of a TPN is independent of the holding times.*

Proof The existence of a non-anticipative tree in $\mathcal{E}(p,n)$ does not depend on the values of the holding times $\sigma_q(m)$, $q \in P$, $m \geq 1$.

6 Constructiveness of the Recursive Equations

In this section, we will see that the recursive equations allow one to recursively determine the variables $V_p(n)$, provided the TPN is live. In other words, whenever the TPN is live, one can use these equations to simulate the dynamic behavior of the system. This property deserves a proof, as these equations might just be a set of implicit equations from which the state of the system cannot be constructed.

In the following, we assume that the TPN is live.

6.1 The Computational Frontline

We first need to introduce the notion of the computational frontline. $\mathcal{F}(i)$ will represent the set of the i first events to occur, that means the i first arrivals of tokens in places during the evolution of the net. Thanks to the construction given in this section, we will see that $\mathcal{F}(i)$ also corresponds to the i first variables $V_p(n)$ that have been computed at the i-th step in the evolution of the computational frontline.

At the beginning, no transition has fired yet. Then, the set of the previous entrances in places is reduced to : $\{V_p(n) \mid n \leq M_p\}$. At the same time, we already know the values $V_p(n) \leq 0$. So the initial computational frontline is $\mathcal{F}(0) = \{(p,n),\ p \in P,\ n \leq M_p\}$. Then, we describe the evolution of the computational frontline iteratively. We will construct $\mathcal{F}(i+1)$ from $\mathcal{F}(i)$.

Assume $\mathcal{F}(i)$ has been constructed. Let $\mathcal{E}_i(p,n)$ denote the set of the Evolution Trees of a couple (p,n) which have all their nodes in $\mathcal{F}(i)$ except perhaps their root (p,n). Finally let

$$W_p^i(n) = \inf_{A \in \mathcal{E}_i(p,n)} \mathcal{C}(A).$$

If $\mathcal{E}_i(p,n)$ is empty, we set $W_p^i(n) = \infty$. It follows from the definition of $W_p^i(n)$ that $W_p^i(n) \geq V_p(n)$.

We introduce a new set $\mathcal{B}(i)$ called the *border* of $\mathcal{F}(i)$:

$\mathcal{B}(i) = \{(p,n) \mid (p,n-1) \in \mathcal{F}(i)\} \cup \{(p,1) \mid p \in P\} \backslash \mathcal{F}(i)$. To an intuitive point of view, $\mathcal{B}(i)$ is the set of the "candidates" for completing $\mathcal{F}(i)$ to $\mathcal{F}(i+1)$.

Theorem 3 *Let $(p,n) \in \mathcal{B}(i)$ such that:*

$$W_p^i(n) = \min_{(q,m)\in\mathcal{B}(i)} W_q^i(m).$$

Then $W_p^i(n) = V_p(n)$.

Proof Assume that, for (p,n) as above, $V_p(n) < W_p^i(n)$. Therefore, if R is a Real Evolution Tree of (p,n), then R is not in $\mathcal{E}_i(p,n)$. Since the net is live, R is a finite tree. Besides, the leaves of R are nodes (q,m) with $m \le M_q$ which are in $\mathcal{F}(i)$. We know that some node of R say (q_1,m_1) is not in $\mathcal{F}(i)$. We consider the subtree of R with root (q_1,m_1). If there is a node (q_2,m_2) in this subtree which is not in $\mathcal{F}(i)$, consider the new subtree with root (q_2,m_2) and repeat this operation. As we know that R is finite and that its leaves are in $\mathcal{F}(i)$ we will stop after a finite number of steps. We eventually get a node (q_k,m_k) now denoted (p_1,n_1) which is not in $\mathcal{F}(i)$ with its subtree in $\mathcal{E}_i(p_1,n_1)$. From the very definition of a RET, we know that this subtree of R is a RET of the node (p_1,n_1), so that its cost is $V_{p_1}(n_1)$. Moreover as this subtree is in $\mathcal{E}_i(p_1,n_1)$, we get, $W_{p_1}^i(n_1) = V_{p_1}(n_1)$. In addition, $V_{p_1}(n_1) \le V_p(n)$, for this is a subtree of R.

If $(p_1,n_1) \in \mathcal{B}(i)$, then $W_p^i(n) \le W_{p_1}^i(n_1) < V_p(n)$. This is a contradiction.

If $(p_1,n_1) \notin \mathcal{B}(i)$, then there exists $m_1 < n_1$ such that $(p_1,m_1) \in \mathcal{B}(i)$. We know that $V_{p_1}(m_1) \le V_{p_1}(n_1) \le V_p(n)$. Therefore $V_{p_1}(m_1) < W_p^i(n) \le W_{p_1}^i(m_1)$. This implies $W_{p_1}^i(m_1) > V_{p_1}(m_1)$.

Let R_1 be a Real Evolution Tree of (p_1,m_1). The preceding strict inequality implies that R_1 is not in $\mathcal{E}_i(p_1,m_1)$. We apply the same method as for the tree R. We get a node (p_2,n_2) that verifies $W_{p_2}^i(n_2) = V_{p_2}(n_2) \le V_{p_1}(n_1) < W_p^i(n)$. So, $(p_2,n_2) \notin \mathcal{B}(i)$. We can find $m_2 < n_2$ with $(p_2,m_2) \in \mathcal{B}(i)$. Using the definition of the variable V_{p_2}, we get $V_{p_2}(m_2) \leq V_{p_2}(n_2) < W_p^i(n) \leq W_{p_2}^i(m_2)$. Therefore, $W_{p_2}^i(m_2) > V_{p_2}(m_2)$. The node (p_2, m_2) has analogous properties to (p_1, n_1). We continue this analysis with R_2 a RET of (p_2, m_2). We will finally get an infinite strictly decreasing sequence $V_{p_1}(m_1) \geq V_{p_2}(m_2) \geq \cdots \geq V_{p_k}(m_k) \geq \cdots$ with all $(p_j, m_j) \in \mathcal{B}(i)$ which is finite. Therefore, there exists two indexes k and l for which $V_{p_k}(m_k) = V_{p_{k+1}}(m_{k+1}) = \cdots = V_{p_l}(m_l)$. In the RET of (p_k, m_k) there is a path between (p_k, m_k) and (p_{k+1}, m_{k+1}). We denote this path by $(p_k, m_k) = (q_1, \mu_1), (q_2, \mu_2), \cdots, (q_{j_1}, \mu_{j_1}) = (p_{k+1}, m_{k+1})$. This path in an evolution tree induces the existence of a directed path in the Petri net between the place p_k and the place p_{k+1}. This path is $q_1, q_2, \cdots, q_{j_1}$. In the same way, we define in the Petri net a directed path between the places p_{k+1} and p_{k+2} denoted $q_{j_1+1}, \cdots, q_{j_2}$. We do the same analysis for all the nodes and we get a cycle in the Petri net : $p_k = q_1, q_2, \cdots, q_{j_l} = p_k$. But on this cycle, every place q_j verifies $\sigma_{q_j}(\mu_j) = 0$ which contradicts the hypothesis on the holding times. Eventually, the first assumption (*i.e.* $W_p^i(n) > V_p(n)$) appears to be false.

We can now set $\mathcal{F}(i+1) = \mathcal{F}(i) \cup \{(p,n)\}$, and be sure that when i goes to infinity, $\mathcal{F}(i)$ increases to $\{(p,n),\ p \in P,\ n \geq 1\}$, since the TPN is live.

6.2 Computation of $V_p(n)$

We can deduce a way to compute every $V_p(n),\ p \in P,\ n \geq 1$ from Theorem 3. Indeed, while $\mathcal{E}(p,n)$ is too complex to be used in any computation, the sets $\mathcal{E}_i(p,n)$ are much more exploitable. Indeed, we can use equations (4.1) to compute $W_p^i(n)$.

At the first step, $\mathcal{F}(0) = \{(p,n) \mid n < M_p\}$ and we set $V_p(n) = v_p(n)\ ,\ \ p \in P,\ n < M_p$.

At step i, assume $\mathcal{F}(i)$ has yet been computed. According to the previous section, we have to compute $W_p^i(n + M_p)$ for all $(p, n + M_p)$ in $\mathcal{B}(i)$. We can achieve this computation without using the integration of the equations (4.1). Indeed we can write:

$$W_p^i(n+M_p) = \tag{6.1}$$

$$\inf_{\left\{ i \in \mathbb{N}^{|\gamma^-(p)|} \;|\; \sum_{t\in\gamma^-(p)} i_t = n \right\}} \max_{\{t\in\gamma^-(p),\, q\in\gamma^-(t)\ \}} \left\{ V_q^i(\eta_{q,t}(i_t)) + \sigma_q(\eta_{q,t}(i_t)) \right\}, \quad n \geq 1,$$

where we set $V_q^i(\eta_{q,t}(i_t)) = +\infty$ if $(q, \eta_{q,t}(i_t)) \notin \mathcal{F}(i)$ and $V_q^i(\eta_{q,t}(i_t)) = V_q(\eta_{q,t}(i_t))$ if $(q, \eta_{q,t}(i_t)) \in \mathcal{F}(i)$.

This equality allows a computation of all the variables $V_p(n)$. Indeed, we compute $\{W_p^i(n)\ ,\ \ (p,n) \in \mathcal{B}(i)\}$ using Equations (6.1). We select the smallest one, say $W_p^i(n)$ and Theorem 3 implies $W_p^i(n) = V_p(n)$. Afterwards, (p,n) is included in the Computational Frontline for step $i+1$ and we repeat the pattern.

This method also allows one to get an enumerable algorithm to establish the liveness of a TPN. As long as the computation of the $W_p^i(n)$ gives a finite smallest value, the net can be assumed to be live. If all the $W_p^i(n)$ are infinite, we have reached a deadlock and the net is not live.

7 Monotony

We define a function $\mathcal{V}$ that associates to a TPN the times $V_p(n)$ and in this section we will study some results of monotony of this function. First we will see that $\mathcal{V}$ is a decreasing function of the initial number of tokens in each place, then that it is an increasing function of the holding times.

In the sake of fitting with practical cases we assume that all transitions are recycled so that the FIFO feature is structurally verified and does not rely on some constraint on the holding times.

7.1 Initial Number of Tokens

The initial tokens in a TPN can model as various things as the lengh of a finite queue in a queueing network, or the presence of customers in the network before it starts. Anyway, the intuition of the acceleration of the net while adding tokens is verified by the following theorem.

Theorem 4 *Let (T, P, Γ) be a live FIFO TPN. We define a new TPN by adding a token in the initial marking in an arbitrary place, say p, anything else remaining unchanged. If we call $V_p'(n)$ the entrance times in the new net, then, we get:*

$$V_q'(m) \leq V_q(m), \quad \forall q \in P, \ m \in IN.$$

Proof The numbering of the initial tokens in the place p of the original net yields $\sigma_p(1) + v_p(1) \leq \sigma_p(2) + v_p(2) \leq \cdots \leq \sigma_p(M_p) + v_p(M_p)$. In the new net we add an initial token with entrance time $v \leq 0$ in this place so that the initial tokens have to be re-ordered such that $\{\sigma_p'(k),\ k = 1, ..., M_p + 1\} = \{\sigma_p(k),\ k = 1..., M_p + 1\}$, $\{v_p'(k),\ 0 \leq k \leq M_p + 1\} = \{v_p(k),\ 0 \leq k \leq M_p\} \cup \{v\}$ and $\sigma_p'(1) + v_p'(1) \leq \sigma_p'(2) + v_p'(2) \leq ... \leq \sigma_p'(M_p+1) + v_p'(M_p+1)$. One can notice that $\sigma_p'(k) + v_p'(k) \leq \sigma_p(k) + v_p(k),\ k = 1, \cdots, M_p$ and $\sigma_p'(M_p + 1) + v_p'(M_p + 1) \leq \sigma_p(M_p + 1) + V_p(M_p + 1)$.

Let R be a Real Evolution Tree in the original net of (q, m). We replace every leaf (p, k) of R with weight $\sigma_p(k) + V_p(k),\ k \in \{1, ..., M_p\}$ by a leaf with weight $\sigma_p'(k) + V_p'(k)$ and every subtree in R with root $(p, M_p + 1)$ by a leaf with weight $\sigma_p'(M_p + 1) + V_p'(M_p + 1)$. We get an Evolution Tree A' of (q, m) in the new net. All the paths passing through the previously modified nodes have their weight decreased thanks to the preceding inequalities. Therefore, the cost of the tree cannot increase. The new TPN is live since the liveness of a TPN is independent of the holding times. Now if R_e' is a Real Evolution Tree of (q, m), in the new net, then we get: $V_q'(m) = \mathcal{C}(R_e') \leq \mathcal{C}(A') \leq \mathcal{C}(R) = V_q(m)$.

Remark 4 This result extends the monotony result obtained in [3] in the particular case of Event Graphs. If the holding and firing times are all random variables defined on some common probability space, Theorem 4 translates into a stochastic monotony of the firing times as a function of the initial marking.

7.2 Holding times

In the next theorem, we see that, as one can guess, the entrance times increase if the holding times increase.

Theorem 5 *Let (P, T, Γ) be a live FIFO TPN with holding times $\sigma_p(n)$ and switching $\rho_{p,t}(n)$. As previously, let $V_p(n)$ denotes the entrance time of the n-th token in place p. We define a new TPN with the same topology, (P, T, Γ) the same switching $\rho_{p,t}(n)$ but with new holding times $\sigma'_p(n)$. $V'_p(n)$ denotes the new entrance times.*

If $\sigma'_p(n) \geq \sigma_p(n), \quad \forall p \in P, \ n \in I\!N$ then $V'_p(n) \geq V_p(n), \quad \forall p \in P, \ n \in I\!N$.

Proof Since the new TPN is live because of the independence of the liveness and of the holding times, let R' be a Real Evolution Tree of (p, n) in the new TPN. We construct an Evolution Tree A of the original TPN replacing the weights of the nodes, $\sigma'_p(n)$, by $\sigma_p(n)$. Thus, the cost of the latter tree is smaller than the cost of the former. Now let R_e be a Real Evolution Tree of (p, n) in the original TPN. We get: $V_p(n) = \mathcal{C}(R_e) \leq \mathcal{C}(A) \leq \mathcal{C}(R') = V'_p(n)$.

Remark 5 This result also extends the monotony result obtained in [3] for Event Graphs, and is related to the one obtained in [14] for the class of Matthes Schemas with deterministic transition kernels. In the stochastic case, we similarly get the stochastic monotony of the firing times as functions of the holding times. A survey of the queueing literature on this type of monotony property can be found in [3]. New queueing result can be added to the set of examples that exhibit this property using Theorem 5, like for instance queueing networks of the Jackson type with i.i.d. routing between the queues when relaxing the exponential assumptions on the holding times.

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